R-mode instability of slowly rotating nonisentropic relativistic stars

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We investigate the properties of *r*-mode instability in slowly rotating relativistic polytropes. Inside the star slow rotation and the low frequency formalism that was mainly developed by Kojima are employed to study axial oscillations restored by the Coriolis force. At the stellar surface, in order to take into account the gravitational radiation reaction effect, we use a near-zone boundary condition instead of the boundary condition usually imposed for asymptotically flat spacetime. Because of the boundary condition, complex frequencies whose imaginary part represents a secular instability are obtained for discrete *r*-mode oscillations in some polytropic models. It is found that such discrete *r*-mode solutions can be obtained only for some restricted polytropic models. The basic properties of the solutions are similar to those obtained by imposing the boundary condition for asymptotically flat spacetime. Our results suggest that the existence of a continuous part of the spectrum cannot be avoided even when its frequency becomes complex due to the emission of gravitational radiation.

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I. INTRODUCTION

Andersson [1] and Friedman and Morsink [2] discovered that all *r* modes, which are quasitoroidal modes mainly restored by the Coriolis force, in all rotating stars become unstable due to the gravitational radiation reaction if other dissipative processes are not considered. This instability is clearly understood by the so called Chandrasekhar-Friedman-Shutz (CFS) mechanism [3–5]. As shown first by Lindblom, Owen, and Morsink [6], this instability still strongly affects the stability of typical neutron star models even if viscous dissipation of the neutron star matter, which tends to stabilize the CFS instability, is taken into account. Since then many studies of oscillation modes restored by the Coriolis force in rotating stars have been done to prove their possible importance in astrophysics (for recent reviews, see, e.g., Refs. [7–10]).

The influence of the *r* mode on the stability of rotating neutron stars is one of the most important and interesting phenomena in astrophysics. In oscillations of neutron stars, the relativistic effect must be important because such stars are sufficiently compact. But most studies have been done within the framework of Newtonian gravity so far, although our understanding of *r* modes has been improved by those investigations. As for *r* modes studied within the framework of general relativity, Kojima [11] derived master equations for *r*-mode oscillation in the lowest order slow rotation approximation and found the possible existence of a continuous part of the spectrum in his equations. Beyer and Kokkotas [12] generally verified the existence of a continuous part of the spectrum in Kojima's equation. Kojima's formalism was developed to include higher order rotational effects by Kojima and Hosonuma [13,14]. Lockitch, Andersson, and Friedman [15] obtained discrete *r*-mode solutions in uniform density stars as well as the continuous part of the spectrum by solving Kojima's equation. Recently, Yoshida [16] and Ruoff and Kokkotas [17] discussed that such discrete *r*-mode solutions are not simply allowed to appear in compressible stellar models. Their results showed that, for typical neutron star models, Kojima's equations do not have such a discrete *r*-mode solution.

These recent developments in understanding of relativistic *r* modes have shown that basic properties of *r*-mode oscillations in relativistic stars are significantly different from those in Newtonian stars. As for nonisentropic stars, most previous studies have shown the existence of a continuous part of the spectrum. This is a great contrast with the Newtonian case. For Newtonian cases, there are discrete mode solutions and no continuous parts of the spectrum for *r* modes in all uniformly rotating stars as long as their rotation velocity is small enough. However, such a drastic change in the behavior of the solutions is not likely to occur due to the inclusion of even a small relativistic effect. Therefore, most authors have considered that a continuous part of the spectrum does not appear if some effects that were omitted in previous studies are taken into account. One such effect is the dissipation effect due to the gravitational radiation reaction. In most studies on *r* modes, a slow motion approximation has been employed in the analysis because of the slow rotation approximation. The slow motion approximation changes the wave type equation into a Laplace type equation. Thus, Kojima's equations do not have solutions with wave character, and the frequency is a real number if asymptotically flat spacetime is assumed. In other words, the influence of gravitational radiation on relativistic *r* modes has not been taken into account so far. In this case, Kojima's equation becomes that of a singular eigenvalue problem for some frequency range, and hence has a continuous part of the spectrum. As suggested by Lockitch *et al.* [15] (see also Ref. [12]), however, Kojima's equation may become that of a regular eigenvalue problem if the frequency has a nonzero imaginary part. Therefore, it is hoped that the existence of a continuous part of the spectrum might be avoided if the effect of the emission of gravitational radiation is considered.

In this paper, accordingly, we will attempt to include the lowest order effect of the gravitational radiation reaction into Kojima's formalism. Because Kojima's equation is not a wave type equation, as mentioned before, we cannot obtain any information about gravitational waves from the equation. In order to include the gravitational radiation reaction effect into Kojima's equation, we will employ a near-zone boundary condition instead of the boundary condition usually used for asymptotically flat spacetime. This boundary condition was introduced by Thorne [18] to include the effect of the gravitational radiation reaction on polar pulsations in a Newtonian star. We start, in Sec. II, with the description of our method of solution. A near-zone boundary condition is introduced there in order to take into account the gravitational radiation reaction. In Sec. III, we show the properties of the r-mode solutions obtained by imposing the boundary condition. Section IV is devoted to discussion and conclusions. Throughout this paper we will use units in which $c=G=1$, where c and G denote the velocity of light and the gravitational constant, respectively.

II. METHOD OF SOLUTION

We consider slowly rotating relativistic stars with a uniform angular velocity Ω, where we take account of the first order rotational effect in Ω. The geometry around the equilibrium stars can be described by the following line element (see, e.g., Ref. [19]):

$$ds^2=-e^{2\nu(r)}dt^2+e^{2\lambda(r)}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2 -2\omega(r)r^2\sin^2\theta dt d\varphi. \tag{2.1}$$

Throughout this paper, the polytropic equation of state is assumed:

$$p=K\rho^{1+1/N}, \tag{2.2}$$

where p and ρ denote the pressure and mass-energy density, respectively. Here N and K are constants.

We consider oscillation modes in rotating relativistic stars such that the eigenfunctions are stationary and are composed of only one axial parity component in the limit $\Omega\to 0$. A subclass of these modes should be the relativistic counterpart of r modes, which are able to oscillate in all slowly rotating Newtonian fluid stars. According to Lockitch *et al.* [15], such modes are allowed to exist only if the star has a nonisentropic structure. Therefore, we assume the stars to be nonisentropic, although the effects due to deviation from isentropic structure on the oscillation modes do not appear in the first order in Ω. According to the formalism by Lockitch *et al.* [15] (see also Ref. [11]), let us write down the pulsation equation for relativistic r modes with accuracy up to the first order in Ω. The metric perturbation $\delta g_{\alpha\beta}$ and the Eulerian changes of the fluid velocity δu^α that do not vanish in the limit $\Omega\to 0$ are given by

$$(\delta g_{t\theta},\delta g_{t\varphi})=ih_{0,l}(r)\left(-\frac{\partial_\varphi Y_{lm}(\theta,\varphi)}{\sin\theta},\sin\theta\partial_\theta Y_{lm}(\theta,\varphi)\right)e^{i\sigma t}, \tag{2.3}$$

$$(\delta u^\theta,\delta u^\varphi)=\frac{iU_l(r)}{r^2}\times\left(\frac{\partial_\varphi Y_{lm}(\theta,\varphi)}{\sin\theta},\frac{\partial_\theta Y_{lm}(\theta,\varphi)}{\sin\theta}\right)e^{i\sigma t}, \tag{2.4}$$

where $Y_{lm}(\theta,\varphi)$ are the usual spherical harmonic functions, and σ denotes the oscillation frequency measured in the inertial frame at spatial infinity. All other perturbed quantities become higher order in Ω. Note that this form of eigenfunction is the same as that for zero-frequency modes in a spherical nonisentropic star, because r modes become zero-frequency ones in the limit $\Omega\to 0$ [20]. The metric perturbation $h_{0,l}$ obeys a second order ordinary differential equation,

$$D_{lm}(r;\bar{\sigma})\left[e^{\nu-\lambda}\frac{d}{dr}\left(e^{-\nu-\lambda}\frac{dh_{0,l}}{dr}\right)-\left(\frac{l(l+1)}{r^2}+\frac{-2+2e^{-2\lambda}}{r^2}+8\pi(p+\rho)\right)h_{0,l}\right]+16\pi(p+\rho)h_{0,l}=0, \tag{2.5}$$

where

$$D_{lm}(r;\bar{\sigma})\equiv 1-\frac{2m\bar{\omega}}{l(l+1)\bar{\sigma}}. \tag{2.6}$$

Here, we have introduced the effective rotation angular velocity of the fluid,

$$\bar{\omega}=\Omega-\omega, \tag{2.7}$$

and the corotating oscillation frequency,

$$\bar{\sigma}=\sigma+m\Omega. \tag{2.8}$$

The velocity perturbation of a fluid U_l is determined from the function $h_{0,l}$ through the following algebraic relation:

$$\left[1-\frac{2m\bar{\omega}}{l(l+1)\bar{\sigma}}\right]U_l+h_{0,l}=0. \tag{2.9}$$

Equations (2.5) and (2.9) are our basic equations, which were derived first by Kojima [11]. Note that Eqs. (2.5) and (2.9) lose their meaning in the $l=0$ case, because there are no axial modes with $l=0$.

Because Eqs. (2.5) are second order ordinary differential equations, two boundary conditions are required to determine solutions uniquely. From the regularity of physical quantities at $r=0$ the function $h_{0,l}$ must vanish at the center of a star. This condition is explicitly given by

$$r\frac{dh_{0,l}}{dr}-(l+1)h_{0,l}=0 \quad \text{as} \quad r\to 0. \tag{2.10}$$

Outside the star, Eqs. (2.5) have general solutions as follows:

$$h_{0,l}(r)=A\sum_{k=0}^{\infty} a_k r^{-l-k}+B\sum_{k=0}^{\infty} b_k r^{l+1+k}, \qquad (2.11)$$

where A and B are arbitrary constants. Here a_k and b_k are constants determined from recurrence relations although the explicit expressions are omitted here. Note that a_k and b_k do not depend on the frequency $\bar{\sigma}$. In most previous studies the condition $B=0$ has been chosen as the boundary condition at spatial infinity because spacetime must be regular everywhere. In this paper we call this condition the "proper boundary condition." This condition means the spacetime is asymptotically flat. Therefore, solutions satisfying the condition $B=0$ cannot describe any gravitational radiation emission from a star at all. In order to include the effect of gravitational radiation emission in the solutions, we must not require zero for B. This was first shown and used in a study of the post-Newtonian approximation by Thorne [18].

Next, let us consider the boundary conditions for quasinormal mode solutions, following the considerations of Thorne [18] (see also Ref. [21]). In the derivation of our basic equations the slow motion approximation is necessarily used as well as the slow rotation one because the frequency of oscillation restored by the Coriolis force is of the same order as the stellar rotation frequency. This slow motion approximation must work nicely near the star as long as low frequency oscillations are considered. Therefore, higher order time derivatives of perturbed quantities are neglected in the governing equations (2.5). If an oscillating star emits gravitational radiation, however, some of such omitted terms must become important in the radiation zone because a term such as σr becomes dominant among all the terms in the governing equations. In addition, rotational effects due to stellar rotation fall off faster than $1/r^2$ as $r\to\infty$. Thus, such a slow motion approximation is not good far from the star, if gravitational waves are radiated from the stars. The Regge-Wheeler equations with correction terms due to the stellar rotation, in fact, govern the axial perturbations sufficiently far from the star even when low frequency oscillations like r modes are induced. Since $\sigma r \gg 1$ in the radiation zone, the Regge-Wheeler equations can be approximately written as

$$r^2\frac{d^2X_l(r)}{dr^2}+\sigma^2r^2X_l(r)-l(l+1)X_l(r)+O\left(\frac{M}{r}\right)=0, \qquad (2.12)$$

where $X_l(r)$ are the Regge-Wheeler functions. Here M denotes the gravitational mass of the star. The metric functions $h_{0,l}$ are determined from the functions X_l by the equation

$$h_{0,l}=\frac{d[rX_l(r)]}{dr}+O\left(\frac{M}{r}\right) \qquad (2.13)$$

(see, e.g., Refs. [22,23]). Here we have considered the limiting case when $M/r \ll 1$ for simplicity. The general solutions to Eqs. (2.12) can be given analytically:

$$X_l(\sigma r)=\sigma r[C\, j_l(\sigma r)+D\, n_l(\sigma r)], \qquad (2.14)$$

where j_l and n_l are spherical Bessel functions and C and D are arbitrary constants. The asymptotic forms in the radiation zone, that is, when $\sigma r \gg 1$, are given by

$$X_l(\sigma r)\sim C\cos\left[\sigma r-\frac{1}{2}(l+1)\pi\right]+D\sin\left[\sigma r-\frac{1}{2}(l+1)\pi\right]. \qquad (2.15)$$

Now we are interested in the quasinormal modes of a star. Thus, the no incoming radiation conditions are required for metric perturbations. From Eq. (2.15) it is found that this condition becomes the relation $D=-i\,C$. For this choice of constants the asymptotic solutions reduce to the form

$$X_l(\sigma r)e^{i\sigma t}\sim C\exp\left[i\sigma(t-r)+\frac{i}{2}(l+1)\pi\right] \quad \text{as} \quad \sigma r\to\infty. \qquad (2.16)$$

Since the frequencies of the oscillations restored by the Coriolis force are proportional to the rotational frequency Ω, $\sigma^2R^2\approx\Omega^2R^2=\epsilon^2M/R$, where ϵ denotes a small parameter for stellar rotation defined as $\epsilon=\Omega/(M/R^3)^{1/2}$. Here, R is the radius of the star. Thus, the surface of the star is approximately in the near zone, that is, $\sigma r \ll 1$, if the stellar rotation is sufficiently slow or the stellar gravity is sufficiently weak. In this approximation, the solutions (2.14) can be written as

$$X_l(\sigma r)\sim -i\,C\frac{(2l-1)!!}{(\sigma r)^l}\left[1+i\frac{(\sigma r)^{2l+1}}{(2l-1)!!(2l+1)!!}\right], \qquad (2.17)$$

where the constraint $D=-iC$ for the no incoming radiation has been used. The approximate solutions above may be valid near the stellar surface because $M/r<1$ is well satisfied at the stellar surface and outside the star for typical neutron star models. Thus, in the near zone, the expressions for metric perturbations $h_{0,l}$ outside the star can be given by Eqs. (2.13) and (2.17) as follows:

$$h_{0,l}\sim C'\frac{1}{r^l}\left[1+i\frac{(l+2)(\sigma r)^{2l+1}}{(l-1)(2l+1)[(2l-1)!!]^2}\right], \qquad (2.18)$$

where C' is an arbitrary constant. As the outer boundary condition we require this solution to connect smoothly to the interior solution obtained by solving Eq. (2.5) at the stellar surface. Thus, the boundary condition is explicitly given by

$$\left[1+i\frac{(l+2)(\sigma R)^{2l+1}}{(l-1)(2l+1)[(2l-1)!!]^2}\right]r\frac{dh_{0,l}}{dr}(R-x)$$
$$+\left[l-i\frac{(l+1)(l+2)(\sigma R)^{2l+1}}{(l-1)(2l+1)[(2l-1)!!]^2}\right]h_{0,l}(R-x)$$
$$=0 \quad \text{as} \quad x\to 0, \qquad (2.19)$$

where we need actual values of the rotation frequency Ω to obtain eigensolutions. In this paper we assume the value of Ω as $\Omega=(\pi\bar{\rho})^{1/2}$, where $\bar{\rho}$ is the average density defined by

$\bar{\rho}=M/(4\pi R^3/3)$. This Ω is an approximate value for the maximum rotation frequency to be possible that settles down uniformly rotating stars in hydrostatic equilibrium. In this paper, we call the condition (2.19) the "near-zone boundary condition." This method is a crude version of matched asymptotic expansions. If we consider a nonrotating limit, that is, the $\sigma=0$ limit, the boundary condition (2.19) becomes an approximation of the proper boundary conditions in which only the lowest order terms in M/R are included. As shown by Lindblom *et al.* [21], a boundary condition similar to this, by using the obtained asymptotic solution (2.18) can give a good approximate value of the eigenfrequency even for *f*-mode oscillation although the *f* mode is not a low frequency oscillation mode.

III. NUMERICAL RESULTS

As shown in previous studies [11,12,15], we should distinguish two cases in treating Eq. (2.5) when the proper boundary condition is imposed. One case is the regular eigenvalue problem and the other is the singular one. For the regular eigenvalue problem Eq. (2.5) may have discrete mode frequencies in the range

$$\frac{2m\bar{\omega}(R)}{l(l+1)}<\bar{\sigma}\leqslant\frac{2m\bar{\omega}(\infty)}{l(l+1)}=\frac{2m\Omega}{l(l+1)}. \tag{3.1}$$

On the other hand, Eq. (2.5) becomes the singular eigenvalue problem if $\bar{\sigma}$ is in the region

$$\frac{2m\bar{\omega}(0)}{l(l+1)}\leqslant\bar{\sigma}\leqslant\frac{2m\bar{\omega}(R)}{l(l+1)}. \tag{3.2}$$

Notice that the range (3.2) is the continuous part of the spectrum of Eq. (2.5). As pointed out by Lockitch *et al.* [15] (see also Ref. [12]), Eq. (2.5) becomes that of the regular eigenvalue problem when the corresponding frequency has a nonzero imaginary part. Thus the frequency ranges above may not have clear mathematical meanings when the frequency has a nonzero imaginary part. According to the frequency ranges above, however, we will divide the eigensolutions into three classes: The first and the second class solutions are characterized by the real parts of their frequencies satisfying inequalities (3.1) and (3.2), respectively. The third class is composed of a compensatory set of the first and second classes.

First of all, we concentrate our attention on *r*-mode solutions with frequencies whose real parts are in the range (3.1). We compute frequencies of mode solutions for several polytropic stellar models. In the present study, only the fundamental *r* modes, whose eigenfunction U_m has no node in the radial direction except at the stellar center, are obtained. This is similar to the result in studies for the proper boundary condition case [16,17]. In Figs. 1 and 2, the real and imaginary parts of the scaled eigenfrequencies $\kappa\equiv\bar{\sigma}/\Omega$ of the *r* modes are, respectively, given as functions of M/R. The eigenfrequencies for stars with four different polytropic indices $N=0$, 0.5, 0.75, and 1 are shown in the different panels in both figures. Only the frequency curves for the modes with $l=m=2$ are depicted in dependence on the relativistic factor M/R because they are considered to be the most important modes for *r*-mode instability.

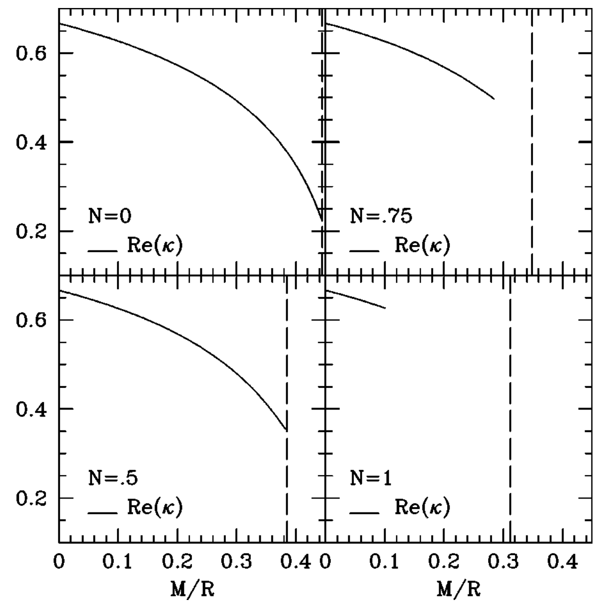

FIG. 1. Real parts of scaled frequencies $\kappa=\bar{\sigma}/\Omega$ of the *r* modes with $l=m=2$ plotted as functions of M/R. In each panel, the frequencies of modes for polytropic models with $N=0$, 0.5, 0.75, and 1 are shown. The labels indicating their polytropic indices N are attached in the corresponding panels. Vertical dotted lines show the maximum values of M/R for equilibrium states: $M/R=0.444$ for $N=0$, $M/R=0.385$ for $N=0.5$, $M/R=0.349$ for $N=0.75$, and $M/R=0.312$ for $N=1.0$.

The real parts of the frequency illustrated in Fig. 1 are in good agreement with Fig. 1 of Ref. [16], in which the fre-

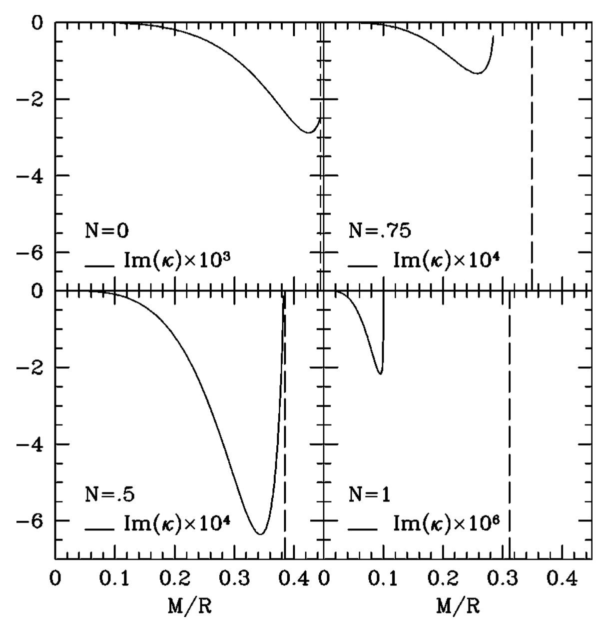

FIG. 2. The same as Fig. 1 but for imaginary parts of frequency.

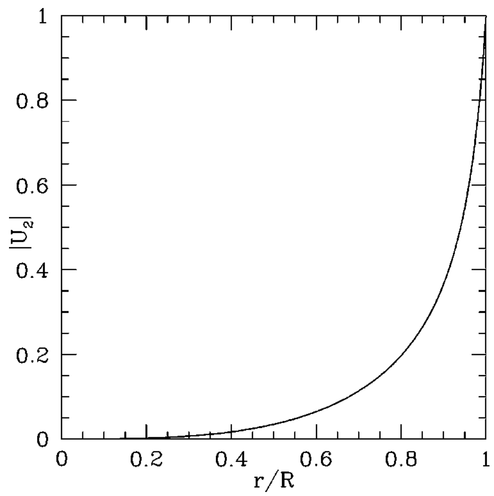

FIG. 3. Absolute value of eigenfunction U_2 of an r mode with $l=m=2$ for an $N=0.5$ polytrope with $M/R=0.1$ is given as a function of r/R, where normalization of the eigenfunction is given by $U_2(R)=1$.

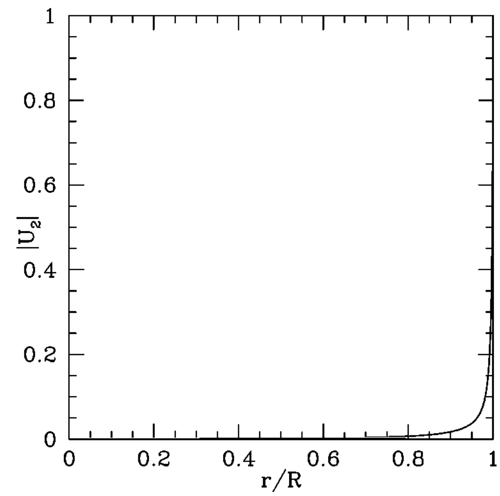

FIG. 4. The same as Fig. 3 but for $M/R=0.37$.

quencies were obtained by imposing the proper boundary condition for asymptotically flat spacetime. The relative differences are less than 0.1%. This shows that our approximation works nicely, and higher order effects of M/r on the outer boundary condition are not so important for the determination of the real parts of the frequency. We also find that the frequency curves in Fig. 1 are terminated at some value of M/R beyond which equilibrium states can still exist. Here, the maximum values of M/R for polytropic equilibrium stars having $N=0$, 0.5, 0.75, and 1.0 are given by 4/9, 0.385, 0.349, and 0.312, respectively. It is also found that the lengths of the frequency curves tend to become shorter as the polytrope index N increases. This feature is similar to that for the case where the proper boundary condition is used. Also, the terminal points of the frequency curves appear at almost the same values of relativistic factors as those for the proper boundary condition case (see Ref. [16]). Beyond the value of the relativistic factors corresponding to those terminal points, we can obtain many eigensolutions with a singular eigenfunction but not with a regular one. Furthermore, the real part of the corresponding frequency belongs to the range (3.2) but not the range (3.1).

From Fig. 2, we can see that the r modes obtained in this study are all unstable. It is also found that the curves for the imaginary parts of κ have one relative minimum near the terminal point of the frequency curves. These minimum values are given by $\mathrm{Im}[\kappa(M/R=0.425)]=-2.9\times10^{-3}$, $\mathrm{Im}[\kappa(M/R=0.344)]=-6.4\times10^{-4}$, $\mathrm{Im}[\kappa(M/R=0.258)]=-1.3\times10^{-4}$, and $\mathrm{Im}[\kappa(M/R=0.095)]=-2.2\times10^{-6}$, for stars having $N=0$, 0.5, 0.75, and 1.0, respectively. The value of $\mathrm{Im}(\kappa)$ also approaches zero as the relativistic factor M/R gets closer to a value corresponding to that for the terminal point of the frequency curves. Those behaviors of $\mathrm{Im}(\kappa)$ can be understood from the distribution of eigenfunctions $U_l(r)$ because $\mathrm{Im}(\kappa)$ is approximately proportional to the square of the current multipole moment. In Figs. 3 and 4 the distributions of the eigenfunction U_m are shown for $N=0.5$ polytropic models having $M/R=0.1$ and $M/R=0.37$, respectively. As seen from these figures the motion of perturbed fluid elements is strongly confined near the stellar surface when the mode frequency becomes closer to the terminal frequency, which satisfies $\bar{\sigma}\approx 2m\bar{\omega}(R)/[l(l+1)]$. We can easily understand this behavior from Eq. (2.9). Consequently, the values of current multipole moments of such modes may become small when the value of the relativistic factor increases. On the other hand, the efficiency of the gravitational radiation emission becomes good with increase of the value of the relativistic factor. Due to both effects above a relative minimum of the imaginary part of frequency may appear. We should note that in the $N=0$ case the value of $\mathrm{Im}(\kappa)$ does not approach zero even when $M/R\sim4/9$, as we can see from Fig. 2. The reason is that the eigenfunctions U_m do not have so strong a peak at the stellar surface even when $M/R\sim4/9$ because the relation $\bar{\sigma}\approx 2m\bar{\omega}(R)/[l(l+1)]$ is not satisfied in this case.

Next let us estimate the instability time scale for the gravitational radiation driven instability of r-mode solutions with frequency in the range (3.1). Here a typical neutron star model whose mass and radius are, respectively, $1.4M_\odot$ and 12.57 km is considered for $N=0$, 0.5, and 0.75 polytropic models. When the star rotates with angular frequency $\Omega=(3/4\times GM/R^3)^{1/2}=8377\ \mathrm{s}^{-1}$, the growth time scales τ_j of the r-mode instability are given by $\tau_j=1.29$ s, 2.04 s, and 2.97 s for the $N=0$, 0.5, and 0.75 models, respectively. Note that for the $N=1$ case the growth time scale cannot be estimated because we cannot find a discrete r-mode solution for that model. These time scales are similar to those obtained from the Newtonian estimate (see, e.g., Ref. [6]).

When we concentrate our attention on solutions whose frequency has a real part in the frequency range (3.2), we obtain a large number of solutions whose eigenfunction has singular behavior in its real part at a radius determined by the solution of $D_{lm}[r;\mathrm{Re}(\bar{\sigma})]=0$. In addition, those solutions have severe truncation errors due to the finite difference method. A similar behavior of solutions appears in oscillations of differentially rotating disks (see, for example, Ref.

[24]). As discussed by Schutz and Verdaguer [24], this is considered a sign of the existence of continuous parts of the spectrum, although the existence has to be proved by other mathematical techniques because an exact continuous spectrum is never obtained from a simple numerical analysis. In the present case, we are sure that the appearance of a continuous spectrum is plausible because the imaginary parts of the frequency are too small to drastically change the character of the solutions derived from the proper boundary condition. Thus, our numerical results suggest the existence of a continuous part of the spectrum in Kojima's equation even when the frequency becomes a complex number. As for regular solutions with frequency in the range (3.2), we cannot obtain such a solution at all. Finally, we consider the third class of solutions, for which the real part of the frequency is in neither the range (3.1) nor (3.2). In this region of real parts of the frequencies, we cannot obtain any solutions at all.

IV. DISCUSSION AND CONCLUSION

In this paper, we have investigated the properties of r-mode instability in slowly rotating relativistic polytropes. Inside the star, slow rotation and the low frequency formalism that was mainly developed by Kojima [11] and Lockitch *et al.* [15] is employed to study axial oscillations restored by the Coriolis force. At the stellar surface, in order to take into account the gravitational radiation reaction effect, we use a near-zone boundary condition, which was devised by Thorne [18] and recently developed for relativistic pulsations by Lindblom *et al.* [21], instead of the usually imposed boundary condition for asymptotically flat spacetime. Due to the boundary condition, complex frequencies whose imaginary part represents secular instability are obtained for r-mode oscillations in some polytropic models. It is found that such discrete r-mode solutions can be obtained only for some restricted polytropic models. The basic properties of mode solutions of Eq. (2.5) that are obtained in this study are similar to those with the boundary condition for asymptotically flat spacetime although their frequency becomes complex because of the near-zone boundary condition.

As suggested by Lockitch *et al.* [15] (see also Refs. [12,16,17]), when an eigenfrequency becomes a complex number, which expresses the damping of the oscillation due to energy dissipation such as the gravitational radiation emission, there is the possibility that the existence of the continuous part of the spectrum in the eigenfrequency of Eq. (2.5) is avoided. In this study, we consider the complex frequency corresponding to the quasinormal mode as the mode solution of Eq. (2.5) by imposing the lowest order near-zone boundary condition. Our numerical results suggest the existence of a continuous part of the spectrum in Kojima's equation even when the frequency is allowed to be a complex number. However, we still think that the existence of a continuous part of the spectrum in axial oscillations restored by the Coriolis force is not plausible because such a property does not appear in Newtonian r modes. The existence of a continuous part of the spectrum in this study might be an artifact due to the approximation because our treatment is the lowest order approximation. In other words, the inclusion of the full effect of gravitational radiation emission might avoid the existence of a continuous part of the spectrum. Another possibility to prevent the appearance of a continuous spectrum might be to avoid a singular reduction in the order of the equation. As Kojima and Hosonuma [14] showed, the basic equations for r-mode oscillations become a fourth order ordinary differential equation for the metric perturbation $h_{0,l}$ when rotational effects up to the third order of $\Omega/(M/R^3)^{1/2}$ are consistently considered. In this equation, the extra two degrees of freedom of solutions may be used to avoid singular behavior of the eigenfunction. Because of these extra boundary conditions, all eigenfrequencies may become discrete. Verification of those possibilities remain for future studies.

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